

A Two-Step Distribution System State Estimator with Grid Constraints and Mixed Measurements

Miguel Picallo^{*†}, Adolfo Anta^{*} and Ara Panosyan^{*}

^{*}General Electric Global Research

Munich, Germany

Email: miguel.picallocruz|anta|panosyan@ge.com

Bart De Schutter[†]

[†]Delft Center for Systems and Control

Delft, The Netherlands

Email: b.deschutter@tudelft.nl

Abstract—In this work we study the problem of State Estimation (SE) in large-scale, 3-phase coupled, unbalanced distribution systems. More specifically, we address the problem of including mixed real-time measurements, synchronized and unsynchronized, from phasor measurement units and smart meters, into existing solutions. We propose a computationally efficient two-step method to update a prior solution using the measurements, while taking into account physical constraint caused by buses with no loads. We test the method on a benchmark test feeder to illustrate the effectiveness of the approach.

I. INTRODUCTION

The operation of power networks requires accurate monitoring of its state: bus voltages, line currents, consumption and generation. This is specially relevant in transmission networks where volatile and distributed generation and consumption cause bidirectional power flow and voltage drops. Normally, State Estimation (SE) consists in estimating the bus voltage phasors and then computing the currents and loads using the power flow equations derived from the structure of the network, represented by the admittance matrix. SE is typically performed by taking several measurements and solving a weighted least squares problem using an iterative approach like Newton-Raphson [1]–[4].

However, SE has not yet been studied and implemented deeply in distribution networks, since they used to have a simple radial structure with a single source bus connected to the grid and injecting power, being the point of common coupling (PCC). Nowadays, this is changing, since the increasing number of sources of distributed generation like PV-panels, electrical vehicles, etc. inject power in the network, causing bidirectional power flow. Consequently, SE becomes necessary in distribution networks. One of the major difficulties for its implementation lies in the structural difference between transmission and distribution networks: in the latter, unbalanced loads and lower X/R ratios cause coupled phases. Therefore, the power flow equations need to be solved for the 3 phases simultaneously, and fast methods like fast decoupled power flow [5] cannot be used. Another limitation is the lack of sufficient real-time measurement units in distribution networks: despite Phasor Measurement Units (PMU) and smart meters are being introduced, their high cost prevent installing the required number of sensors to make the system fully

observable. Some methods have been proposed to perform SE by exploiting the network structure and nature of the measurements, such as branch-current-based SE [6]–[8]. Other methods use the few measurements available to improve the SE accuracy of previous methods based on load assumptions only [9]–[11]. Yet, most of this work is limited to transmission networks, small-scale distribution networks, or single-phase networks. Some recent work extends it more general networks, but assumes measurement availability at every bus [12].

Our main contributions can be summarized as follows: First, we include equality constraints more efficiently by a dimension reduction. Then, we extend some recently proposed methods that split the problem [9], to large-scale, 3-phase coupled, unbalanced and constrained distribution systems, with linear as well as nonlinear real-time measurements considering different sources of measurement noises.

The rest of the paper is structured as follows: Section II presents some background about power networks. Section III discusses the different types of measurements. Section IV presents the standard methodologies for general SE. Section V proposes the linear dimension reduction method and extends recent methodologies. Section VI defines the test case simulation and shows its results. Finally, Section VII presents the conclusions and proposed future work.

II. DISTRIBUTION SYSTEM MODEL

A distribution system can be modeled as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with nodes $\mathcal{V} = \{1, \dots, N_{\text{bus}}\}$, edges $\mathcal{E} = \{(v_i, v_j) \mid v_i, v_j \in \mathcal{V}\}$ and edge weights \mathcal{W} . The nodes represent the buses in the network, where power is injected or consumed, the edges the electrical connections in the grid, and their weights express the impedance between the buses, which is determined by the length and type of the line cables.

In 3-phase networks buses may have up to 3 phases, so that the voltage V_i at bus i lives in \mathbb{C}^{P_i} , where P_i is the number of its phases. Moreover, $V_{\text{source}} \in \mathbb{C}^3$ are the known voltages at the source bus with reference angles $\angle V_{\text{source}} = [0 \ \frac{-2\pi}{3} \ \frac{2\pi}{3}]^T$. The state of the network is then typically represented with the vector of the remaining bus voltages $V \in \mathbb{C}^N$ where N depends on the number of buses and phases per bus.

The impedance of the connection between buses i and j is a square symmetric matrix $Z_{i,j} \in \mathbb{C}^{P_{i,j} \times P_{i,j}}$ where $P_{i,j}$ is the number of phases in the connection $i \rightarrow j$. Since phases are coupled, $Z_{i,j}$ has off-diagonal terms different from 0. These impedance matrices are used to compute the admittance matrix

$Y \in \mathbb{C}^{(N+3) \times (N+3)}$, which defines the structure of the system. Assuming lines with no shunt admittance, the element of Y corresponding to the connection between phase l of bus i and phase k of j can be expressed as:

$$(Y)_{i_l, j_k} = \begin{cases} -\frac{1}{(Z_{i,j})_{l,k}} & \text{if } i \neq j \\ \sum_{m \neq i} \frac{1}{(Z_{i,m})_{l,k}} & \text{if } i = j \end{cases} \quad (1)$$

where i_l is the index of phase l of bus i , similarly for j_k , $(\cdot)_{i,j}$ denotes the i, j element of a matrix, and $(Y)_{i_l, j_k} = 0$ if no connection exists between these phases of these buses. Using the admittance matrix Y , the formulas of the power flow equations to compute the current I and the load S at all buses are expressed as:

$$\begin{bmatrix} I_{\text{source}} \\ I \end{bmatrix} = Y \begin{bmatrix} V_{\text{source}} \\ V \end{bmatrix} = \begin{bmatrix} Y_a & Y_b \\ Y_c & Y_d \end{bmatrix} \begin{bmatrix} V_{\text{source}} \\ V \end{bmatrix} \quad (2)$$

$$S = \text{diag}(\bar{I})V$$

where $\bar{(\cdot)}$ denotes the complex conjugate, $\text{diag}(\cdot)$ represents the diagonal operator, converting a vector into a diagonal matrix. Separating Y in blocks according to the indices of the source bus V_{source} , see (2), the voltage V for the non-source buses can be rewritten as:

$$\begin{aligned} V &= Y_d^{-1}I - Y_d^{-1}Y_c V_{\text{source}} \\ &= Y_d^{-1}(\text{diag}(\bar{V}))^{-1}\bar{S} - Y_d^{-1}Y_c V_{\text{source}} \end{aligned} \quad (3)$$

so that the voltage under no load ($I = 0, S = 0$) is:

$$V_0 = -Y_d^{-1}Y_c V_{\text{source}} \quad (4)$$

III. MEASUREMENTS

Several different sources of information are available to solve the SE problem:

- 1) *Pseudo-measurements*: Load estimations for every hour, based on predictions and known installed load capacity at every bus, and represented by S_{pseudo} . Since they are estimations rather than actual measurements, their noise is modeled with a relative large standard deviation (a typical value is $\sigma_{\text{pseudo}} \approx 50\%$):

$$S = S_{\text{pseudo}} + \Re\{S_{\text{pseudo}}\}\omega_{\text{pseudo},\Re} + j\Im\{S_{\text{pseudo}}\}\omega_{\text{pseudo},\Im}$$

with $\omega_{\text{pseudo},\Re}, \omega_{\text{pseudo},\Im} \sim N(0, \sigma_{\text{pseudo}}^2 I_{d,N})$, where $\Re\{\cdot\}, \Im\{\cdot\}$ denote the real and imaginary part, and $I_{d,N}$ the identity matrix with size $N \times N$. The noise covariance of these measurements is then:

$$\begin{aligned} \Sigma_{S_{\text{pseudo}}} &= \mathbb{E}[(S - S_{\text{pseudo}})(S - S_{\text{pseudo}})^*] \\ &= \sigma_{\text{pseudo}}^2 \text{diag}(|S_{\text{pseudo}}|^2) \end{aligned} \quad (5)$$

where $\mathbb{E}[\cdot]$ denotes expectation, $(\cdot)^*$ complex conjugate transpose, $|\cdot|$ element-wise magnitudes of a complex vector and $(\cdot)^2$ element-wise square.

- 2) *Virtual measurements*: Buses with no loads connected, acting only as a connection [13]. They can be modeled as physical constraints for the voltage states. Let $\varepsilon = \{i, \dots, j\}$ be the set with their indices, then:

$$(S)_{\varepsilon} = 0, (I)_{\varepsilon} = 0 \quad (6)$$

where $(\cdot)_{\varepsilon}$ denotes the elements at indices in ε .

- 3) *Real-time measurements*: Voltages and currents measurements from PMU, smart meters, or conventional remote terminal units. There are two kinds: GPS-synchronized

measuring magnitude and phase angle, and unsynchronized for only magnitude. We model the noises with a low standard deviation for magnitude and angle, $\sigma_{\text{mag}} \approx 1\%$ and $\sigma_{\text{ang}} \approx 0.01\text{rad}$ respectively, according to the IEEE standard for PMU [14].

Synchronized measurements can be expressed using a linear approximation (see Appendix) with magnitude and angle noise caused by the measurements and by an imperfect synchronization. For a number of N_{measL} measurements $z_{\text{measL}} \in \mathbb{C}^{N_{\text{measL}}}$ we have:

$$z_{\text{measL}} \approx C_{\text{measL}}V + \text{diag}(C_{\text{measL}}V)(\omega_{\text{mag}} + j\omega_{\text{ang}}) \quad (7)$$

where $\omega_{\text{mag}}, \omega_{\text{ang}} \sim N(0, \sigma_{\text{meas}}^2 I_{d,N_{\text{measL}}})$ using $\sigma_{\text{meas}} = \sigma_{\text{ang}} = \sigma_{\text{mag}}$ (since they are similar [14]), and C_{measL} is the matrix mapping state voltages to measurements. For element j of $C_{\text{measL}}V$ measuring at phase l of bus i :

$$(C_{\text{measL}}V)_j = (C_{\text{measL}})_{j,\bullet}V = \begin{cases} V_{i_l} & \text{for a voltage measurement} \\ (Y)_{i_l,\bullet}V & \text{for a current measurement} \\ (Y)_{i_l,m_l}(V_{i_l} - V_{m_l}) & \text{for a branch-current } i \rightarrow m \text{ measurement} \end{cases} \quad (8)$$

where $(\cdot)_{j,\bullet}$ denotes row j .

Unsynchronized measurements have a nonlinear relation $C_{\text{measNL}}(V)$ with the voltages V . For N_{measNL} measurements $z_{\text{measNL}} \in \mathbb{R}^{N_{\text{measNL}}}$:

$$z_{\text{measNL}} = C_{\text{measNL}}(V) + \text{diag}(C_{\text{measNL}}(V))\omega_{\text{measNL}} \quad (9)$$

with $\omega_{\text{measNL}} \sim N(0, \sigma_{\text{meas}}^2 I_{d,N_{\text{measNL}}})$. For element j of $C_{\text{measNL}}(V)$ measuring at phase l of bus i :

$$(C_{\text{measNL}}(V))_j = \begin{cases} |V_{i_l}| & \text{for a voltage measurement} \\ |(Y)_{i_l,\bullet}V| & \text{for a current measurement} \\ |(Y)_{i_l,m_l}(V_{i_l} - V_{m_l})| & \text{for a branch-current } i \rightarrow m \text{ measurement} \end{cases} \quad (10)$$

Since the measurement noises are small, their covariance matrices can be approximated using the measurements:

$$\begin{aligned} \Sigma_{\text{measL}} &= 2\sigma_{\text{meas}}^2 \text{diag}(|C_{\text{measL}}V|^2) \approx 2\sigma_{\text{meas}}^2 \text{diag}(|z_{\text{measL}}|^2) \\ \Sigma_{\text{measNL}} &= \sigma_{\text{meas}}^2 \text{diag}(C_{\text{measNL}}(V)^2) \approx \sigma_{\text{meas}}^2 \text{diag}(z_{\text{measNL}}^2) \end{aligned}$$

IV. STANDARD METHODOLOGY FOR STATE ESTIMATION

In a power network, SE consists in estimating the state of the network, represented by the bus voltages vector V . The standard methodology for SE computes the maximum likelihood estimation by solving a constrained nonlinear weighted least squares problem with all measurements and estimations: $(S_{\text{pseudo}})_{\varepsilon^c}, z_{\text{measL}}, z_{\text{measNL}}$, their noise covariance and the system constraints $(S_{\text{pseudo}})_{\varepsilon} = 0$, where ε^c denotes the complementary of ε . Typically this is solved using the Newton-Raphson method [1]. However, since this requires the use of gradients and the power flow formulas in (2) are not holomorphic, i.e. complex differentiable, the problem is solved in real variables using a polar representation $V_{\text{polar}} = [|V|^T \angle V^T]^T$:

$$\min_{V_{\text{polar}}} \|x - h(V_{\text{polar}})\|_{W^{-1}} \quad \text{s.t. } g(V_{\text{polar}}) = 0 \quad (11)$$

where $\|x\|_A^2 = x^*Ax$ is the norm with respect to the real positive-definite matrix of weights A , z is the vector of mea-

measurements in rectangular coordinates and $h(\cdot)$ is a nonlinear function mapping V_{polar} to measurements:

$$z = \begin{bmatrix} \Re\{S_{\text{pseudo},\varepsilon^c}\} \\ \Im\{S_{\text{pseudo},\varepsilon^c}\} \\ \Re\{z_{\text{measL}}\} \\ \Im\{z_{\text{measL}}\} \\ z_{\text{measNL}} \end{bmatrix}, h(V_{\text{polar}}) = \begin{bmatrix} \Re\{S_{\varepsilon^c}(V(V_{\text{polar}}))\} \\ \Im\{S_{\varepsilon^c}(V(V_{\text{polar}}))\} \\ \Re\{(C_{\text{measL}}V(V_{\text{polar}}))\} \\ \Im\{(C_{\text{measL}}V(V_{\text{polar}}))\} \\ C_{\text{measNL}}(V(V_{\text{polar}})) \end{bmatrix}$$

where $S(V)$ represents S as a function of V , $V(V_{\text{polar}})$ is defined similarly. The function $g(\cdot)$ indicates the null loads:

$$g(V_{\text{polar}}) = [\Re\{S_{\varepsilon}(V(V_{\text{polar}}))\}^T \Im\{S_{\varepsilon}(V(V_{\text{polar}}))\}^T]^T$$

and the weight matrix W^{-1} is the inverse of the measurement noises in rectangular variables

$$W = \begin{bmatrix} \sigma_{\text{pseudo}}^2 \text{diag} \left(\begin{bmatrix} \Re\{(S_{\text{pseudo}})_{\varepsilon^c}\}^2 \\ \Im\{(S_{\text{pseudo}})_{\varepsilon^c}\}^2 \end{bmatrix} \right) & 0 & 0 \\ 0 & \Sigma_{\text{rect,measL}} & 0 \\ 0 & 0 & \Sigma_{\text{measNL}} \end{bmatrix}$$

where 0 denotes a matrix with 0 entries, and

$$\Sigma_{\text{rect,measL}} \approx \sigma_{\text{meas}}^2$$

$$\begin{bmatrix} \text{diag}(|z_{\text{measL}}|^2) & 2\text{diag}(\Re\{z_{\text{measL}}\}\Im\{z_{\text{measL}}\}) \\ 2\text{diag}(\Re\{z_{\text{measL}}\}\Im\{z_{\text{measL}}\}) & \text{diag}(|z_{\text{measL}}|^2) \end{bmatrix}$$

This problem (11) is typically solved iteratively using Lagrangian multipliers λ [15], so that the size of the optimization variables is given by $[V_{\text{polar}}^T \lambda]^T \in \mathbb{R}^{2N+2|\varepsilon|}$.

V. PROPOSED METHODS FOR STATE ESTIMATION

In this section we propose new SE methods that significantly improve the standard methodology in (11). First, we present a method that simplifies the problem and reduces its complexity, while obtaining an equal solution; then, we propose another method that significantly reduces the computation cost while achieving approximately the same accuracy.

A. Weighted least squares in subspace

Previous methods either do not consider constraints, or do it by introducing Lagrange multipliers as in (11), which increases the complexity of the algorithm. Here we propose an alternative method to include the constraints using a dimension reduction through a linear transformation that produces feasible solutions. The space of feasible solutions is:

$$\begin{aligned} \{V \mid I_{\varepsilon}(V) &= (Y_d)_{\varepsilon}V + (Y_c)_{\varepsilon}V_{\text{source}} = 0\} \\ &= \{V \mid \exists x \in \mathbb{C}^{N-|\varepsilon|} \text{ s.t. } V = Fx + V_p\} \end{aligned} \quad (12)$$

where V_p satisfies $I_{\varepsilon}(V_p) = 0$, and F is such that:

$$(Y_d)_{\varepsilon}F = 0 \quad (13)$$

Therefore, V_0 is chosen for V_p , and F is built using an orthonormal basis of the kernel of $(Y_d)_{\varepsilon}$ e.g. by computing the Singular Value Decomposition of $(Y_d)_{\varepsilon} = U\mathcal{S}\mathcal{V}^*$, and taking the columns of \mathcal{V} with null singular value. Then, $F = \ker((Y_d)_{\varepsilon})$ and $F^*F = I_d$, where $\ker(\cdot)$ denotes the kernel subspace. If we consider the rectangular representation of voltages $V_{\text{rect}} = [\Re\{V\}^T \Im\{V\}^T]^T$ instead of the polar

representation V_{polar} , the subspace equations can still be represented linearly in real variables:

$$\begin{aligned} \{V_{\text{rect}} \mid \Re\{I_{\varepsilon}(V_{\text{rect}})\} &= 0, \Im\{I_{\varepsilon}(V_{\text{rect}})\} = 0\} \\ &= \left\{ V_{\text{rect}} \mid \begin{bmatrix} \Re\{(Y_d)_{\varepsilon}\} & -\Im\{(Y_d)_{\varepsilon}\} \\ \Im\{(Y_d)_{\varepsilon}\} & \Re\{(Y_d)_{\varepsilon}\} \end{bmatrix} V_{\text{rect}} + \begin{bmatrix} \Re\{(Y_c)_{\varepsilon}V_{\text{source}}\} \\ \Im\{(Y_c)_{\varepsilon}V_{\text{source}}\} \end{bmatrix} = 0 \right\} \\ &= \{V_{\text{rect}} \mid \exists \tilde{x} \in \mathbb{C}^{2N-2|\varepsilon|} \text{ s.t. } V_{\text{rect}} = \tilde{F}\tilde{x} + \tilde{V}_p\} \end{aligned} \quad (14)$$

where now

$$\tilde{V}_p = \begin{bmatrix} \Re\{V_0\} \\ \Im\{V_0\} \end{bmatrix}, \tilde{F} = \ker \left(\begin{bmatrix} \Re\{(Y_d)_{\varepsilon}\} & -\Im\{(Y_d)_{\varepsilon}\} \\ \Im\{(Y_d)_{\varepsilon}\} & \Re\{(Y_d)_{\varepsilon}\} \end{bmatrix} \right) \quad (15)$$

Then the problem in (11) becomes:

$$V_{\text{rect}} = \tilde{F} \left(\arg \min_x \|z - \tilde{h}(\tilde{F}\tilde{x} + \tilde{V}_p)\|_{W^{-1}} \right) + \tilde{V}_p \quad (16)$$

where $\tilde{h}(\cdot)$ is $h(\cdot)$ in rectangular coordinates, using $V(V_{\text{rect}})$ instead of $V(V_{\text{polar}})$, so that $\tilde{h}(V_{\text{rect}}) = h(V_{\text{polar}})$. This again can be solved using the Newton-Raphson method:

$$\tilde{x}_{k+1} = \tilde{x}_k + \Delta\tilde{x}_k$$

with

$$\Delta\tilde{x}_k = (H_k^T W^{-1} H_k)^{-1} H_k^T W^{-1} (z - \tilde{h}(\tilde{F}\tilde{x}_k + \tilde{V}_p))$$

$$H_k = \nabla_{\tilde{x}} \tilde{h}(\tilde{F}\tilde{x} + \tilde{V}_p) |_{\tilde{x}_k} = \nabla_{V_{\text{rect}}} \tilde{h}(V_{\text{rect}}) |_{V_{\text{rect},k}} \tilde{F}$$

Note that now no Lagrangian multipliers are required. We eliminate the equality constraints by embedding the solution into a smaller subspace of feasible solutions. This simplifies the method and reduces its computational cost, since we reduce the size of the optimization variables $\tilde{x} \in \mathbb{R}^{2N-2|\varepsilon|}$.

B. Power flow solution plus optimal linear update

The methods (11) and (16) are iterative, and thus can become computationally expensive as the size of the network N increases. This may be a limitation in large distribution systems when fast calculations are required every few seconds. Therefore, we propose an alternative method splitting the problem into two parts as in [9]: first, we use only the constraints and the load estimates S_{pseudo} available beforehand to solve the power flow problem offline and to obtain a prior solution defined as V_{prior} ; then, we update in real-time the solution using the measurements $z_{\text{measL}}, z_{\text{measNL}}$ to compute a better estimate, represented by V_{post} . We also prove that several alternatives for the update are equal under certain conditions.

1) Obtaining a prior solution:

a) The prior estimate can be computed using the previous method in (16) without real-time measurements, so that $z = [\Re\{S_{\text{pseudo},\varepsilon^c}\}^T \Im\{S_{\text{pseudo},\varepsilon^c}\}^T]^T$ and $\tilde{h}(V_{\text{rect}}) = [\Re\{S_{\varepsilon^c}(V(V_{\text{rect}}))\}^T \Im\{S_{\varepsilon^c}(V(V_{\text{rect}}))\}^T]^T$. This method will estimate real and imaginary parts $V_{\text{rect,prior}}$, the estimation error covariance of which can be approximated by:

$$\Sigma_{\text{rect,prior}} \approx \tilde{F} (H_k^T W_{\text{rect}}^{-1} H_k)^{-1} \tilde{F}^T \quad (17)$$

The covariance matrix in complex variables is:

$$\begin{aligned} \Sigma_{\text{prior}} &= \Sigma_{\text{rect,prior},\Re\Re} + \Sigma_{\text{rect,prior},\Re\Im} \\ &\quad - j\Sigma_{\text{rect,prior},\Re\Im} + j\Sigma_{\text{rect,prior},\Im\Im} \end{aligned} \quad (18)$$

where $\Sigma_{\text{rect,prior},\Re\Re}, \Sigma_{\text{rect,prior},\Re\Im}$ indicate the covariance of real and imaginary parts respectively, and $\Sigma_{\text{rect,prior},\Re\Im}$ the covariance between real and imaginary parts.

- b) An alternative approach using complex variables would be the method proposed in [9]. This method can be extended to a 3-phase coupled, unbalanced system using the voltage V_0 under no loads defined in (4). There, V_0 is used as initial value in this iterative method, where the iterations are inspired by (3), i.e. for the k -iteration we have:

$$V_{k+1} = Y_d^{-1}(\text{diag}(\bar{V}_k))^{-1}\bar{S} + V_0 \quad (19)$$

The estimation error covariance can be approximated by the covariance after the first iteration, which corresponds to the linear approximation in [16]:

$$\Sigma_{\text{prior}} \approx Y_d^{-1}(\text{diag}(\bar{V}_0))^{-1}\Sigma_{S_{\text{pseudo}}}(\text{diag}(V_0))^{-1}(Y_d^{-1})^*$$

Remark 1. The estimated current at any iteration k is:

$$I_{k+1} = (\text{diag}(\bar{V}_k))^{-1}\bar{S} \quad (20)$$

If the loads satisfy $(S)_\varepsilon = 0$, then $(I_{k+1})_\varepsilon = 0$. This means that there exists a vector $x \in \mathbb{C}^{N-|\varepsilon|}$ s.t. $V_k = Fx + V_0$ and thus V_k is feasible $\forall k$ w.r.t. (12). Note that we use $F \in \mathbb{C}^{N \times (N-|\varepsilon|)}$ instead of \bar{F} since $V_k \in \mathbb{C}^N$.

Remark 2. This method can be extended to real variables to get Σ_{prior} in rectangular coordinates:

$$V_{\text{rect},k+1} = B_k \begin{bmatrix} \Re\{S_{\text{pseudo}}\} \\ \Im\{S_{\text{pseudo}}\} \end{bmatrix} + V_{\text{rect},0} \quad (21)$$

with

$$B_k = \begin{bmatrix} \Re\{Y_d^{-1}\} & -\Im\{Y_d^{-1}\} \\ \Im\{Y_d^{-1}\} & \Re\{Y_d^{-1}\} \end{bmatrix} \left(\text{diag} \left(\begin{bmatrix} |V_k|^2 \\ |V_k|^2 \end{bmatrix} \right) \right)^{-1} \cdot \begin{bmatrix} \text{diag}(\Re\{V_k\}) & \text{diag}(\Im\{V_k\}) \\ \text{diag}(\Im\{V_k\}) & -\text{diag}(\Re\{V_k\}) \end{bmatrix}$$

and estimation error covariance:

$$\Sigma_{\text{rect},\text{prior}} = B_0 \sigma_{S_{\text{pseudo}}}^2 \text{diag} \left(\begin{bmatrix} \Re\{S_{\text{pseudo}}\}^2 \\ \Im\{S_{\text{pseudo}}\}^2 \end{bmatrix} \right) B_0^T \quad (22)$$

The advantage of approach b) w.r.t. a) is that b) can be solved in complex variables. In both cases, since the load estimates S_{pseudo} are known beforehand, this part of the problem can be computed offline, taking this iterative computational cost outside the real-time operation of the system.

2) Updating with only synchronized measurements:

If there are not any unsynchronized measurements, the mapping from the state variables to the real-time measurements (8) is linear and thus holomorphic in $x \in \mathbb{C}^{N-|\varepsilon|}$. Consequently, the update can be solved using complex variables to obtain a linear expression. Different alternative methods are possible, here we present two of them and later show their equivalence:

- a) If the prior solution V_{prior} is feasible, then there exists a vector x_{prior} s.t. $V_{\text{prior}} = Fx_{\text{prior}} + V_0$. It can be computed as $x_{\text{prior}} = F^*(V_{\text{prior}} - V_0)$, with estimation error covariance $F^*\Sigma_{\text{prior}}F$, or $x_{\text{rect},\text{prior}} = \tilde{F}^T(V_{\text{rect},\text{prior}} - V_{\text{rect},0})$ for real variables. Then the maximum likelihood update can be computed using weighted least squares:

$$V_{\text{post}} = V_0 + F \arg \min_x \left\| \begin{bmatrix} x - x_{\text{prior}} \\ C_{\text{measL}}(Fx + V_0) - z_{\text{measL}} \end{bmatrix} \right\|_{W_{\text{post}}^{-1}}^2 \quad (23)$$

where the optimal weights W_{post}^{-1} are:

$$W_{\text{post}} = \begin{bmatrix} F^*\Sigma_{\text{prior}}F & 0 \\ 0 & \Sigma_{\text{measL}} \end{bmatrix} \quad (24)$$

Taking derivatives at (23) we obtain the solution:

$$V_{\text{post}} = F((F^*\Sigma_{\text{prior}}F)^{-1} + F^*C_{\text{measL}}^*\Sigma_{\text{measL}}^{-1}C_{\text{measL}}F)^{-1} \cdot ((F^*\Sigma_{\text{prior}}F)^{-1}x_{\text{prior}} + F^*C_{\text{measL}}^*\Sigma_{\text{measL}}^{-1} \cdot (z_{\text{measL}} - C_{\text{measL}}V_0)) + V_0 \quad (25)$$

Remark 3. Considering the first order (and thus linear) approximation of the measurement function $\tilde{h}(\cdot)$, in [10] is proven that solving the weighted least squares problem in one step (16) and in two steps: weighted least squares for power flow plus update linear (23), produce the same exact solutions. Consequently, we can expect a similar accuracy between the one step and the two steps methods.

- b) An alternative proposed [9] is to improve the solution using an minimum variance linear update:

$$V_{\text{post}} = V_{\text{prior}} + K(z_{\text{measL}} - C_{\text{measL}}V_{\text{prior}}) \quad (26)$$

Then the error covariance of estimation V_{post} is:

$$\Sigma_{\text{post}} = \Sigma_{\text{prior}} + K(\Sigma_{\text{measL}} + C_{\text{measL}}\Sigma_{\text{prior}}C_{\text{measL}}^*)K^* - KC_{\text{measL}}\Sigma_{\text{prior}} - \Sigma_{\text{prior}}C_{\text{measL}}^*K^* \quad (27)$$

and the optimal gain K is computed minimizing the expected error $\mathbb{E}[(V_{\text{post}} - V)^*(V_{\text{post}} - V) | V_{\text{prior}}, z_{\text{measL}}] = \mathbb{E}[\text{tr}(\Sigma_{\text{post}}) | V_{\text{prior}}, z_{\text{measL}}]$:

$$K = \arg \min_K \mathbb{E}[\text{tr}(\Sigma_{\text{post}}) | V_{\text{prior}}, z_{\text{measL}}] = \Sigma_{\text{prior}}C_{\text{measL}}^*(C_{\text{measL}}\Sigma_{\text{prior}}C_{\text{measL}}^* + \Sigma_{\text{measL}})^{-1} \quad (28)$$

Proposition 1. If V_{prior} is a feasible solution to (12), as the solutions in Section V-B1, then the solution (26) satisfies (12).

Proof. For V_{prior} : $(Y_d)_\varepsilon V_{\text{prior}} + (Y_c)_\varepsilon V_{\text{source}} = 0$. Also, since the first term in K is F , because K starts with $\Sigma_{\text{prior}} = F^*\Sigma_{x_{\text{prior}}}F^*$, we have $(Y_d)_\varepsilon K = 0$ due to (13). Therefore, V_{post} from (26) satisfies (12):

$$\begin{aligned} & (Y_d)_\varepsilon V_{\text{post}} + (Y_c)_\varepsilon V_{\text{source}} \\ &= (Y_d)_\varepsilon V_{\text{prior}} + (Y_c)_\varepsilon V_{\text{source}} + (Y_d)_\varepsilon K(z_{\text{measL}} - C_{\text{measL}}V) \\ &= 0 + 0 = 0 \end{aligned} \quad \blacksquare$$

Proposition 2. If V_{prior} is a feasible solution to (12), then both alternatives (26) and (25) are equal.

Proof. We start from expression (25), using Woodbury's identity [17] on the inverse term and (28) for F we get:

$$\begin{aligned} V_{\text{post}} &= F((F^*\Sigma_{\text{prior}}F) - (F^*\Sigma_{\text{prior}}F)F^*C_{\text{measL}}^* \\ &\quad \cdot (C_{\text{measL}}\Sigma_{\text{prior}}C_{\text{measL}}^* + \Sigma_{\text{measL}})^{-1}C_{\text{measL}}F(F^*\Sigma_{\text{prior}}F)) \\ &\quad \cdot ((F^*\Sigma_{\text{prior}}F)^{-1}x_{\text{prior}} + F^*C_{\text{measL}}^*\Sigma_{\text{measL}}^{-1} \\ &\quad \cdot (z_{\text{measL}} - C_{\text{measL}}V_0)) + V_0 \\ &= (F(F^*\Sigma_{\text{prior}}F) - KC_{\text{measL}}F(F^*\Sigma_{\text{prior}}F)) \\ &\quad \cdot ((F^*\Sigma_{\text{prior}}F)^{-1}x_{\text{prior}} + F^*C_{\text{measL}}^*\Sigma_{\text{measL}}^{-1} \\ &\quad \cdot (z_{\text{measL}} - C_{\text{measL}}V_0)) + V_0 \\ &= V_0 + (I_d - KC_{\text{measL}})Fx_{\text{prior}} \\ &\quad + (I_d - KC_{\text{measL}})F(F^*\Sigma_{\text{prior}}F)F^*C_{\text{measL}}^*\Sigma_{\text{measL}}^{-1} \\ &\quad \cdot (z_{\text{measL}} - C_{\text{measL}}V_0) \end{aligned} \quad (29)$$

Doing some manipulation to the formula of K in (28) we get:

$$(I_d - KC_{\text{measL}})\Sigma_{\text{prior}}C_{\text{measL}}^*\Sigma_{\text{measL}}^{-1} = K \quad (30)$$

So that finally we can convert the expression in (25) to (26):

$$\begin{aligned} V_{\text{post}} &= V_0 + (I_d - KC_{\text{measL}})Fx_{\text{prior}} + K(z_{\text{measL}} - C_{\text{measL}}V_0) \\ &= V_{\text{prior}} + K(z_{\text{measL}} - C_{\text{measL}}V_{\text{prior}}) \end{aligned} \quad \blacksquare$$

The maximum likelihood estimator of weighted least square coincides with the minimum variance estimator of the linear update, since we consider gaussian distributions for the noise.

3) Updating with unsynchronized measurements:

If there are nonlinear measurements, since the magnitude of a complex number as in (10) is not an holomorphic function, the update step needs to be solved using real variables. It becomes:

$$V_{\text{rect,post}} = V_{\text{rect,prior}} + \tilde{K} \begin{bmatrix} \Re\{z_{\text{measL}} - C_{\text{measL}}V(V_{\text{rect,prior}})\} \\ \Im\{z_{\text{measL}} - C_{\text{measL}}V(V_{\text{rect,prior}})\} \\ z_{\text{measNL}} - C_{\text{measNL}}(V(V_{\text{rect,prior}})) \end{bmatrix} \quad (31)$$

Given that

$$\begin{bmatrix} \Re\{C_{\text{measL}}V(V_{\text{rect,prior}})\} \\ \Im\{C_{\text{measL}}V(V_{\text{rect,prior}})\} \end{bmatrix} = \begin{bmatrix} \Re\{C_{\text{measL}}\} - \Im\{C_{\text{measL}}\} \\ \Im\{C_{\text{measL}}\} \quad \Re\{C_{\text{measL}}\} \end{bmatrix} V_{\text{rect,prior}}$$

and defining

$$\begin{aligned} C_{\text{measLNL}} &= \begin{bmatrix} \Re\{C_{\text{measL}}\} - \Im\{C_{\text{measL}}\} \\ \Im\{C_{\text{measL}}\} \quad \Re\{C_{\text{measL}}\} \end{bmatrix} \\ \Sigma_{\text{measLNL}} &= \begin{bmatrix} \Sigma_{\text{rect,measL}} & 0 \\ 0 & \Sigma_{\text{measNL}} \end{bmatrix} \end{aligned}$$

The first-order approximation of the estimation error covariance of this nonlinear filter is:

$$\begin{aligned} \Sigma_{\text{rect,post}} &\approx \tilde{K}(\Sigma_{\text{measLNL}} + C_{\text{measLNL}}\Sigma_{\text{rect,prior}}C_{\text{measLNL}}^T)\tilde{K}^T \\ &+ \Sigma_{\text{rect,prior}} - \tilde{K}C_{\text{measLNL}}\Sigma_{\text{rect,prior}} - \Sigma_{\text{rect,prior}}C_{\text{measLNL}}^T\tilde{K}^T \end{aligned} \quad (32)$$

where now the optimal gain \tilde{K} is:

$$\tilde{K} = \Sigma_{\text{rect,prior}}C_{\text{measLNL}}^T(C_{\text{measLNL}}\Sigma_{\text{rect,prior}}C_{\text{measLNL}}^T + \Sigma_{\text{measLNL}})^{-1}$$

Remark 4. As in Proposition 2, it can be proven that the first-order approximation of the estimation error covariance (32) of (31) is equal to the one approximating the estimation error covariance of (23) including nonlinear measurements, computed similarly as in (17).

VI. SIMULATION RESULTS

A simulation of 24 hours with 15 min intervals is run on a test case to compare the different methods for SE. Here we describe the settings of this simulation and analyze the results.

A. Settings

- **System:** We use the 123-bus test feeder [18] available online [19], see Fig. 1. This is a challenging example, since it is 3-phase coupled, unbalanced, and larger than other examples in the literature [7], [9].
- **Measurements** (see Fig. 1): Voltage measurements (red solid, circle for phasor, square for magnitude only) are placed at buses 79, 95, 83 and 300, current measurements (blue dashed, circle and square) at bus 65 and 48, and branch current measurements (blue dashed arrow) at branch 150 (after the regulator) \rightarrow 149. The standard deviation used to simulate noise in the measurements is $\sigma_{\text{meas}} = 0.01$ [14].

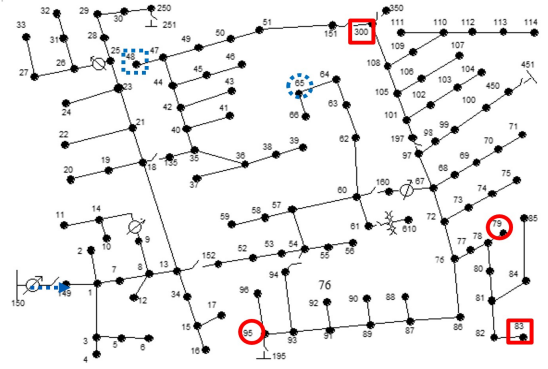


Fig. 1: 123-bus test feeder with measurements location. The network image has been taken from [19].

This sensors locations are chosen since they correspond to nodes with big loads and/or at the end of a feeder.

- **Load Profiles:** Historical data from a German local utility is scaled so that the mean load is the base load of the 123-bus test feeder and the relative standard deviation is $\sigma_{\text{pseudo}} = 50\%$ at every hour. The values of S_{pseudo} are an average of the bus loads for every hour.
- **Metric:** For every simulation time step t , a normalized root mean square error (nRMSE) is determined for the voltages estimated by every method j :

$$\text{nRMSE}_{t,V_{\text{method}_j}} = \frac{1}{|V_{\text{base}}|} \sqrt{\frac{1}{N} \sum_{i=1}^N |V_{\text{method}_j,t,i} - V_{t,i}|^2}$$

where $V_{t,i}$, $V_{\text{method}_j,t,i}$ denote the i -element at time t of the actual voltage and the voltage estimated by method j respectively. This method quantifies the percentage deviation of the solutions relative to the network's base voltage V_{base} .

- **Nomenclature:** The solutions compared are denoted as: V_{WLS} , V_{WLSNL} , which correspond to the solutions of (16) without and with nonlinear measurements respectively; V_{prior} corresponding the prior solution (19) before the update; and V_{post} , V_{postNL} corresponding the updated solutions (23), (31).

B. Results

Fig. 2a shows how a solution with real-time measurements like V_{post} clearly outperforms the prior solution V_{prior} using only load estimations. This shows that even when few measurements are available, in this case only 4 measurements in a 123-bus network, using them can dramatically increase the accuracy of the prediction. Moreover, Fig. 2b shows that the two-step solutions V_{post} and V_{postNL} perform statistically as well as the respective weighted least squares solutions V_{WLS} and V_{WLSNL} . However, the methods using weighted least squares are computationally much more expensive than the others, specially the one with nonlinear measurements V_{WLSNL} , see Fig. 3 for the comparison of different computation times. These algorithms are coded in Python and run on an Intel Core i7-5600U CPU at 2.60GHz with 16GB of RAM.

To summarize, the solutions including real-time measurements improve the accuracy of the estimation, specially if adding extra nonlinear measurements. However, the two-step approach

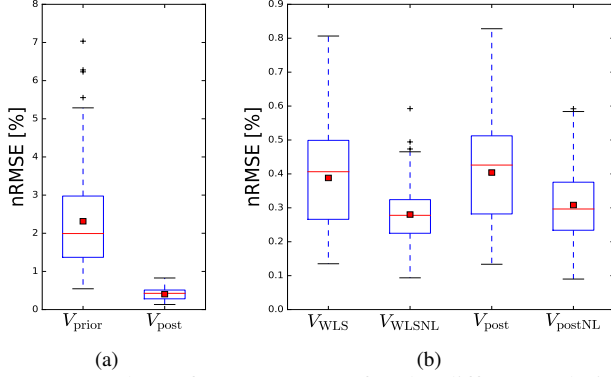


Fig. 2: Box-plots of MAPE errors for the different solutions. The red line indicates the median, the red square the mean.

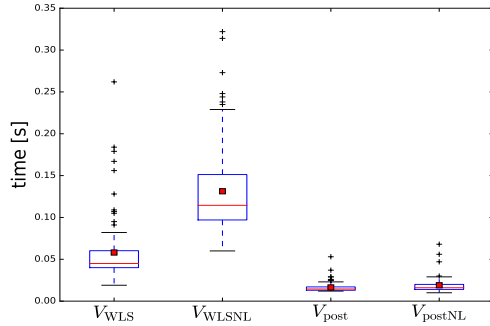


Fig. 3: Box-plots of execution times for the different solutions. The red line indicates the median, the red square the mean.

has a smaller computational cost, and remains small after adding nonlinear measurements.

VII. CONCLUSIONS

We have proposed a method for state estimation in large-scale, 3-phase coupled, unbalanced and constrained distribution systems, with mixed measurements: phasor-synchronized (linear) and magnitude-unsynchronized (nonlinear), and different sources of noise. We have shown that the proposed method is as accurate as standard methods, but computationally more efficient in real-time operation, since it is a non-iterative method. The computational cost reduction is achieved by splitting the problem into an offline and an online problem. The iterative offline method estimates the prior solution, taking most of the computational cost, while the online problem updates this solution in a single step.

Future work could include adding different sources of distributed generation and investigating how they affect the solutions; developing methodologies to compute the minimum and optimal allocation of measurement units to guarantee a given accuracy level; and adding dynamic state equations to exploit the historical information provided by the sensors.

APPENDIX

A PMU measure of a complex number $u = |u| e^{j\theta_u}$ with magnitude and angle noise can be expressed as:

$$\begin{aligned} \tilde{u} &= (|u| + \omega_{\text{mag}}) e^{j(\theta_u + \omega_{\theta})} \\ &= (|u| + \omega_{\text{mag}})(\cos(\theta_u + \omega_{\theta}) + j \sin(\theta_u + \omega_{\theta})) \end{aligned}$$

where $\omega_{\text{mag}} \sim N(0, 1\%|u|)$ (or $\frac{1}{|u|}\omega_{\text{mag}} \sim N(0, 0.01)$), $\omega_{\theta} \sim N(0, 0.01\text{rad})$, according to the standards in [14]. Using trigonometric identities for $\cos(\theta_u + \omega_{\theta})$ and $\sin(\theta_u + \omega_{\theta})$; using approximations $\sin(\omega_{\theta}) \approx \omega_{\theta}$ and $\cos(\omega_{\theta}) \approx 1$, since $|\omega_{\theta}| \ll 1$; and neglecting second order terms of noise $O(\omega_{\theta}\omega_{\text{mag}})$, then we have:

$$\begin{aligned} \tilde{u} &\approx |u|(\cos(\theta_u) + j \sin(\theta_u)) + \omega_{\text{mag}}(\cos(\theta_u) + j \sin(\theta_u)) \\ &\quad + \omega_{\theta}|u|(-\sin(\theta_u) + j \cos(\theta_u)) \\ &= u + \omega_{\text{mag}}\frac{1}{|u|}u + j\omega_{\theta}u = u + u\left(\frac{1}{|u|}\omega_{\text{mag}} + j\omega_{\theta}\right) \end{aligned}$$

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